Semiclassical approach to quantum Loschmidt echo in deep quantum regions: from validity to breakdown

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(Dated: September 4, 2012)

In this paper, we study validity of semiclassical predictions for the decay of quantum Loschmidt echo in deep quantum regions. This is done numerically in several models (the chaotic regime in the sawtooth model and in the kicked rotator model, as well as the critical region of a 1D Ising chain in transverse field), in which appropriate effective Planck constants can be introduced. In particular, we study the change from validity to breakdown of the semiclassical predictions, when the effective Planck constants are increased. Our numerical simulations show that the semiclassical predictions may work even when the effective Planck constant is not small, in particular, in the so-called Fermi-Golden-rule regime.

PACS numbers: 03.65.Ta; 03.65.-w; 03.65.Yz

I. INTRODUCTION

The semiclassical theory is powerful in dealing with many problems in various fields of physics [1, 2]. It is usually expected to work in the semiclassical regime, in which the (effective) Planck constant is sufficiently small in certain relative sense. An interesting question is how deep in the quantum regime could semiclassical predictions remain valid. Obviously, the answer should depend on the physical quantity of interest.

In this paper, we study a quantity, to which the semiclassical approach has recently been found quite successful in the semiclassical regime. It is the so-called quantum Loschmidt echo (LE) [3–5], which is given by the overlap of the time evolution of the same initial state under two slightly different Hamiltonians,

$$M(t) = |m(t)|^2 \text{ with}$$
$$m(t) = \langle\Psi_0| \exp(iH_1 t/\hbar) \exp(-iH_0 t/\hbar)|\Psi_0\rangle, \quad (1)$$

where $H_1 = H_0 + \epsilon V$ with ϵ a small quantity and V a generic perturbation. The quantity $m(t)$ is usually called the amplitude of the LE. The LE gives a measure to the stability of quantum motion under small perturbation.

The LE has quite rich behaviors, depending on the nature of the dynamics of the underlying classical system, as well as on the perturbation strength. Usually, the LE has a quadratic decay within certain initial time interval, as predicted by the first-order perturbation theory [6]. After the initial time interval, in a chaotic system, roughly speaking, the LE has a Gaussian decay [5, 7, 8] below a perturbative border and has an exponential decay in the so-called Fermi-golden-rule (FGR) regime above the perturbative border with intermediate perturbation strength [9–14]. With further increasing perturbation strength, in the so-called Lyapunov regime under relatively strong perturbation, the LE usually has a perturbation-independent decay [11, 14–19]; but, in certain cases, a perturbation-dependent oscillation in the decay rate may also appear [14, 18–20].

On the other hand, in regular systems, in the case of one degree of freedom, the LE has a Gaussian decay [10], followed by a power-law decay [21, 22]. Meanwhile, in the case of many-degrees of freedom, the LE usually has an exponential decay for times not very long [23].

The above discussed semiclassical predictions for the LE have been tested in the deep semiclassical regime, with sufficiently small effective Planck constants. Here, we are interested in the extent to which they may still be valid in the opposite deep-quantum regime, when effective Planck constants are not so small. To achieve this goal, it is necessary to mainly rely on numerical simulations in concrete models. Our numerical simulations in the sawtooth model and the kicked rotor model show that the semiclassical predictions may work well even in the deep quantum regime.

We also study the LE decay in the vicinity of the quantum phase transition (QPT) of a 1D Ising chain in transverse field. At a QPT, manifested in level crossing of the ground level with other level(s), the ground state has drastic change(s) in its fundamental properties [24]. The properties of QPT have drawn wide attention and have been studied extensively in the recent years. In addition to traditional quantities employed in the study of phase transitions, quantities borrowed from the quantum information field have also been found useful in characterizing QPT, such as fidelity as the overlap of ground states [25–27] and the LE [28–31]. In the neighborhood of the critical point of the Ising chain, an effective Planck constant may be introduced and, as shown in Ref.[23], the semiclassical theory may be used to predict the decaying behavior of the survival probability, which is a special case of the LE. In this paper, we give further investigation, in particular, we study validity of the semiclassical predictions when the effective Planck constant is not small.

The paper has the following structure. In Sec.II, we recall the semiclassical approach to the LE decay. In

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Sec.III, we study the LE decay in the deep-quantum region of the sawtooth model and of the kicked rotator model. Section IV is denoted to the study of the LE decay in the vicinity of the QPT of the 1D Ising chain in transverse filed. Finally, conclusions and discussions are given in Sec.V.

II. SEMICLASSICAL APPROACH TO THE LE

In this section, we recall semiclassical predictions for the decay of LE. As well known, the quantum transition amplitude from a point $\mathbf{r}_0$ to a point $\mathbf{r}$ in a d-dimensional configuration space within a time period t can be expressed in terms of Feynman's path integral [32, 33]. In the semiclassical limit, one may use the stationary phase approximation to approximately compute the transition amplitude. Contributions from paths close to classical trajectories give the following well-known semiclassical evolution, in terms of Van Vleck-Gutzwiller propagator $K_{sc}(\mathbf{r};\mathbf{r}_0;t)$,

$$\Psi_{sc}(\mathbf{r};t) = \int d\mathbf{r}_0 K_{sc}(\mathbf{r};\mathbf{r}_0;t)\Psi_0(\mathbf{r}_0), \tag{2}$$

where $K_{sc}(\mathbf{r};\mathbf{r}_0;t) = \sum_s K_s(\mathbf{r};\mathbf{r}_0;t)$ and

$$K_s(\mathbf{r};\mathbf{r}_0;t) = \frac{C_s^{1/2}}{(2\pi i\hbar)^{d/2}} \exp\left[\frac{i}{\hbar}S_s(\mathbf{r};\mathbf{r}_0;t) - \frac{i\pi}{2}\mu_s\right]. \tag{3}$$

Here, the subscript s indicates classical trajectory, $C_s^{1/2} = |\det(\partial^2 S_s/\partial r_{i0}\partial r_j)|$, μ_s is the Maslov index counting conjugate points, and $S_s(\mathbf{r};\mathbf{r}_0;t)$ is the action, i.e., the time integral of the Lagrangian along the trajectory s, $S_s(\mathbf{r};\mathbf{r}_0;t) = \int_0^t dt'\mathcal{L}$.

Let us consider an initial narrow Gaussian wave packet,

$$\Psi_0(\mathbf{r}_0) = \left(\frac{1}{\pi\xi^2}\right)^{d/4} \exp\left[\frac{i}{\hbar}\tilde{\mathbf{p}}_0 - \frac{(\mathbf{r}_0-\tilde{\mathbf{r}}_0)^2}{2\xi^2}\right], \tag{4}$$

where $(\tilde{\mathbf{r}}_0, \tilde{\mathbf{p}}_0)$ indicates the packet center and ξ is the dispersion. Semiclassically, the LE is written as $M_{sc}(t) = |m_{sc}(t)|^2$, where

$$m_{sc}(t) = \int d\mathbf{r} \left[\Psi_{sc}^{H_1}(\mathbf{r};t)\right]^* \Psi_{sc}^{H_0}(\mathbf{r};t). \tag{5}$$

As shown in Refs.[15, 34], the amplitude $m_{sc}(t)$ has the following explicit expression,

$$m_{sc}(t) \simeq \left(\frac{\xi^2}{\pi\hbar^2}\right)^{\frac{d}{2}} \int d\mathbf{p}_0 \exp\left[\frac{i}{\hbar}\Delta S(\mathbf{p}_0,\tilde{\mathbf{r}}_0;t) - \frac{(\mathbf{p}_0-\tilde{\mathbf{p}}_0)^2}{(\hbar/\xi)^2}\right], \tag{6}$$

where $\Delta S(\mathbf{p}_0,\tilde{\mathbf{r}}_0;t)$ is the action difference along two nearby trajectories in two systems H_1 and H_0. In the first-order classical perturbation theory, with the difference between the two trajectories neglected, one has

$$\Delta S(\mathbf{p}_0,\tilde{\mathbf{r}}_0;t) \simeq \epsilon \int_0^t dt' V[\mathbf{p}_0(t')]. \tag{7}$$

The LE amplitude in Eq.(6) can be written as an integration over ΔS, hence, the same for the LE,

$$M(t) = \left|\int d\Delta S e^{i\Delta S/\hbar} P(\Delta S)\right|^2, \tag{8}$$

where $P(\Delta S)$ is the distribution of the action difference defined by

$$P(\Delta S) \simeq \left(\frac{\xi^2}{\pi\hbar^2}\right)^{\frac{d}{2}} \int d\mathbf{p}_0 \delta[\Delta S - \Delta S(\mathbf{p}_0,\tilde{\mathbf{r}}_0;t)] \exp\left[-\frac{(\mathbf{p}_0-\tilde{\mathbf{p}}_0)^2}{(\hbar/\xi)^2}\right]. \tag{9}$$

In a chaotic system, averaged over initial states, the distribution $P(\Delta S)$ is usually close to a Gaussian distribution. When the perturbation is not strong, in the so-called FGR regime, ΔS are relatively small and deviation of $P(\Delta S)$ from the Gaussian distribution is not large. In this case, the semiclassical theory predicts the following FGR decay for the LE,

$$M_{sc}(t) \simeq e^{-2\sigma^2 R(E)t}, \tag{10}$$

where $\sigma = \epsilon/\hbar$ and $R(E)$ is the classical action diffusion constant[9],

$$R(E) = \int_0^\infty dt \langle (V[r(t)] - \overline{V})(V[r(0)] - \overline{V})\rangle. \tag{11}$$

With increasing perturbation strength, deviation of the distribution $P(\Delta S)$ from the Gaussian form can not be neglected and one enters into the so-called Lyapunov regime. In this regime, when the time is not short, ΔS close to stationary points give the main contribution to the LE and the stationary phase approximation predicts the following perturbation-independent decay [18, 19],

$$M_{sc}(t) \propto \exp\left[-\Lambda_1(t)t\right], \tag{12}$$

where

$$\Lambda_1(t) = -\frac{1}{t}\lim_{\delta x(0)\to 0} \ln \overline{\left|\frac{\delta x(t)}{\delta x(0)}\right|^{-1}}, \tag{13}$$

with average taken over initial states. One should note that $\Lambda_1(t)$ is usually not equal to the Lyapunov exponent λ_L,

$$\lambda_L = \lim_{t\to\infty}\frac{1}{t}\lim_{\delta x(0)\to 0} \ln\left|\frac{\delta x(t)}{\delta x(0)}\right|, \tag{14}$$

due to local fluctuations. When the time t is sufficiently long, the LE has a decay determined by the long time limit of $\Lambda_1(t)$ discussed in Ref.[16]. In a system with a homogeneous phase space, i.e., with constant local Lyapunov exponent, $\Lambda_1(t)$ is given by the Lyapunov exponent and the LE has the Lyapunov decay [15].

In a 1D regular system with periodic classical motion, the LE has the following semiclassical expression [22],

$$M_{sc}(t) \simeq c_0(1+\xi^2 t^2)^{-1/2} e^{-\Gamma t^2/(1+\xi^2 t^2)}, \quad (15)$$

where $c_0 \sim 1$ and

$$\Gamma = \frac{1}{2}\left(\frac{\varepsilon w_p}{\hbar}\frac{\partial U}{\partial p_0}\right)^2, \quad \xi = \left|\frac{\epsilon w_p^2}{2\hbar}\frac{\partial^2 U}{\partial p_0^2}\right|, \quad (16)$$

with the derivatives evaluated at the center of the initial Gaussian wave packet. Here, w_p is the width of the initial Gaussian wave packet in the momentum space, $U = \frac{1}{T_p}\int_0^{T_p} V\, dt$ and T_p is the period of the classical motion.

On the other hand, in a regular system with many degrees of freedom, when the time is not long, due to the large number of frequencies, the system has a motion similar to a chaotic one (though the motion shows quasi-periodicity for sufficiently long times). As a result, the LE has a FGR-type decay in Eq.(10) when the time is not very long [23], with

$$R(E) = \frac{1}{2t}\left(\left\langle\left[\int_0^t V(t)dt\right]^2\right\rangle - \left\langle\int_0^t V(t)dt\right\rangle^2\right). \quad (17)$$

Finally, we note that Eq.(8) is a general expression, not restricted to the semiclassical limit. To show this point, one may use Feynman's path-integral formulation. For brevity, lets us write Feynman's propagator as

$$K_F(\mathbf{r}, \mathbf{r}_0; t) = \mathcal{N}\sum_\alpha \exp\{iS_\alpha(\mathbf{r}, \mathbf{r}_0; t)/\hbar\}, \quad (18)$$

where α indicates possible paths going from $\mathbf{r}_0$ to $\mathbf{r}$ within a time interval t and $\mathcal{N}$ is the normalization coefficient. Using this propagator, the exact time evolution of the wave function $\Psi(\mathbf{r}, t)$ can be written in a form similar to that in Eq.(2), with K_{sc} replaced by K_F. Then, substituting the expression obtained into the definition of $m(t)$ in Eq.(1), one obtains

$$m(t) = \mathcal{N}\mathcal{N}'\int d\mathbf{r}_0\mathbf{r}_0' \sum_{\alpha\alpha'} \exp(i\Delta S_e/\hbar)\Psi_0(\mathbf{r}_0)\Psi_0^*(\mathbf{r}_0'), \quad (19)$$

where $\Delta S_e = S_\alpha^{H_0} - S_{\alpha'}^{H_1}$. It is seen that the LE amplitude $m(t)$ can always be written as an integration over the exact action difference ΔS_e, with the distribution $P(\Delta S)$ defined accordingly, as a result, the LE can always be written in the form of Eq.(8).

III. LE DECAY IN DEEP QUANTUM REGION OF TWO KICKED SYSTEMS

For the purpose of studying validity of semiclassical predictions in the deep quantum region, it would be convenient to employ models in which effective Planck constants can be suitably introduced. In such a model, the value of effective Planck constant gives a natural measure for quantum "deepness".

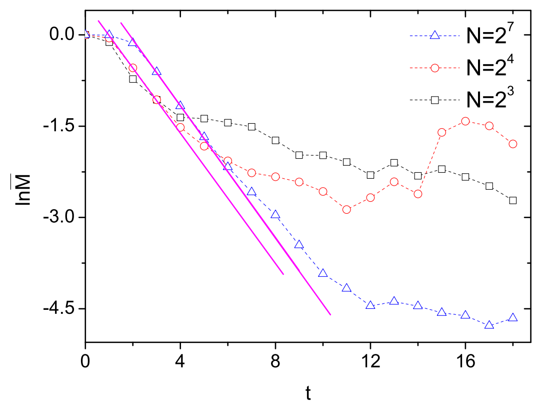

FIG. 1: LE decay in the FGR regime in the sawtooth model, for different values N of the dimension of the Hilbert space. Parameters: $K = 2.0$ and $\sigma = 0.5$. The LE (averaged over initial states) has two stages of decay. In the first stage, it follows the semiclassically-predicted FGR decay (solid straight lines), namely, $e^{-\sigma^2\pi^4 t/45}$ in Eq.(27); in the second stage, it decays with slower rates.

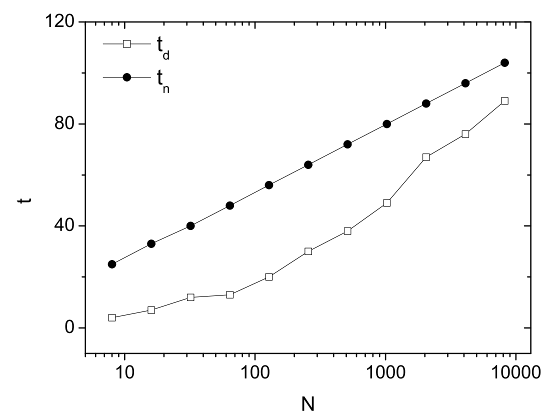

FIG. 2: Variation of t_d (empty squares) with N in the sawtooth model, where t_d is the time at which the second-stage decay of the LE in the FGR regime appears (see Fig.1). For comparison, we also plot t_n (solid circles), the time at which the FGR decay is expected to reach the saturation value $1/N$, i.e., $t_n = (45\ln(N))/(\sigma^2\pi^4)$. Parameters: $K = 2.0$, $\sigma = 0.2$.

A. Two kicked models

We employ the sawtooth model and the kicked rotator model, whose Hamiltonians have the following form,

$$H = \frac{1}{2}p^2 + V(r)\sum_{n=0}^{\infty}\delta(t - nT), \quad (20)$$

where $V(r) = -k(r-\pi)^2/2$ for the sawtooth model and $V(r) = k\cos r$ for the kicked rotator model. Here, for

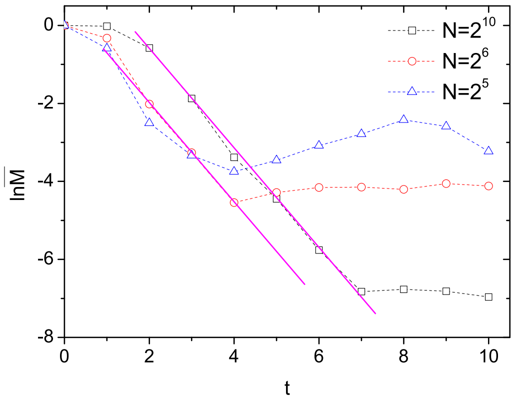

FIG. 3: Similar to Fig.1, but for $\sigma = 3.0$ in the Lyapunov regime of the sawtooth model. The LE has a decay close to the semiclassically-predicted Lyapunov decay, namely, $e^{-\lambda_L t}$ (solid lines), for $N \geq 2^6$.

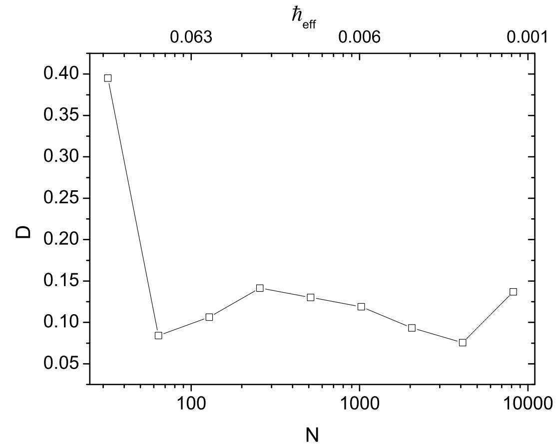

FIG. 4: Variation of the deviation D in Eq.(29) with N in the Lyapunov regime of the sawtooth model, with parameters $K = 2.0$ and $\sigma = 3.0$. The value of D keeps small for $N \geq N_c = 64$ and becomes large when N is smaller than N_c.

simplicity in discussion, we consider their dimensionless form. Hereafter, we take unit Planck constant, $\hbar = 1$.

The classical dynamics in the kicked rotator model generates the standard map,

$$\begin{aligned} \tilde{p}_{n+1} &= \tilde{p}_n + K\sin(r_n) \quad (\text{mod } 2\pi), \\ r_{n+1} &= r_n + \tilde{p}_{n+1} \quad (\text{mod } 2\pi), \end{aligned} \tag{21}$$

where $\tilde{p}_n = Tp_n$, $K = kT$. The classical motion is regular for sufficiently small K and is almost chaotic for K larger than 6 or so. In the sawtooth model, one has the following classical mapping,

$$\begin{aligned} \tilde{p}_{n+1} &= \tilde{p}_n + K(r_n - \pi) \quad (\text{mod } 2\pi), \\ r_{n+1} &= r_n + \tilde{p}_{n+1} \quad (\text{mod } 2\pi). \end{aligned} \tag{22}$$

Equation.(22) can be written in the matrix form

$$\begin{pmatrix} \tilde{p}_{n+1} \\ r_{n+1} - \pi \end{pmatrix} = \begin{pmatrix} 1 & K \\ 1 & K+1 \end{pmatrix} \begin{pmatrix} \tilde{p}_n \\ r_n - \pi \end{pmatrix}, \tag{23}$$

The constant matrix in the above equation possesses two eigenvalues $1 + (K \pm \sqrt{(K)^2 + 4K})/2$. The motion of the classical sawtooth model is completely chaotic for $K > 0$, with the Lyapunov exponent

$$\lambda_L = \ln(\{2 + K + [(2 + K)^2 - 4]^{1/2}\}/2), \tag{24}$$

given by the larger eigenvalue of the constant matrix.

We utilize the method of quantization on torus to get the quantum versions of the above two classical systems, with periodic boundary condition for the coordinate and momentum variables, $0 \leq r < r_m$, $0 \leq p < p_m$ [1, 35–37]. For a Hilbert space with dimension N, an effective Planck constant can be introduced, denoted by h_{eff}, with $h_{\text{eff}} = r_m p_m / N$. In the specific choice of $r_m = p_m = 2\pi$, which will be taken in what follows, we have $\hbar_{\text{eff}} = 2\pi/N$. The value of $\hbar_{\text{eff}}$ gives a measure for the "deepness" in the quantum region, with relatively-large value of $\hbar_{\text{eff}}$ implying being relatively deep in the quantum region. The evolution operator for one period of time T, with $T = 2\pi/N = \hbar_{\text{eff}}$, is written as

$$U = \exp\left[-\frac{i}{2\hbar_{\text{eff}}}\tilde{p}^2\right] \exp\left[-\frac{i}{\hbar_{\text{eff}}}\tilde{V}(r)\right], \tag{25}$$

where $\tilde{V}(r) = TV(r)$.

In the two kicked models discussed above, the quantity $R(E)$ appearing in the FGR decay in Eq.(10) has the following expression [38, 39],

$$R(E) = \frac{1}{2}C(0) + \sum_{l=1}^{\infty} C(l), \tag{26}$$

where $C(l) = \langle\{V[r(l)] - \langle V\rangle\}\{V[r(0)] - \langle V\rangle\}\rangle$. In the sawtooth model with an integer K, $C(0) = \pi^4/45$ and $C(l) = 0$ for $l \neq 0$, hence,

$$M_{sc}(t) \simeq e^{-\pi^4\sigma^2 t/45}. \tag{27}$$

Meanwhile, in the kicked rotator model, $R(E)$ is a function of the parameter K and does not have an explicit analytical expression.

B. Numerical results in sawtooth model

In this subsection, we discuss our numerical simulations obtained in the sawtooth model. In the FGR regime, when N is not large, it was found that, beyond some initial times, the LE has two stages of decay (see Fig.1): In the first stage, the LE follows the semiclassically-predicted FGR decay, while, in the second stage, it is somewhat slower than the FGR decay.

After these two stages of decay, the LE oscillates around its saturation value, which is on average approximately equal to $1/N$ [10]. Interestingly, the first-stage decay of the LE exists even for small dimension N of the Hilbert space, hence, for large value of the effective Planck constant.

We use t_d to indicate the transition time of the above-discussed two stages of decay of the LE, i.e., the time at which obvious deviation from the FGR decay appears. As seen in Fig.2, t_d increases with increasing N. In addition, we observe that the second-stage decay of the LE approaches the FGR decay with increasing N, that is, the difference between the decay rates in the two stages decreases.

To get further understanding in the above-discussed first and second-stage decay of the LE, let us reconsider the expression of the LE in Eq.(8). The distribution $P(\Delta S)$ always has some deviation from its Gaussian approximation, which we denote by P_G with G standing for Gaussian, that is,

$$P(\Delta S) = P_G + \Delta P. \tag{28}$$

The above discussed numerical results imply that the deviation ΔP is not sufficiently large for times shorter than t_d (beyond some initial times), as a result, the LE still follows the FGR decay. However, for times beyond t_d, the deviation can not be neglected.

As discussed previously, Eq.(8) is not just a semiclassical expression, but, is an exact expression, if the distribution $P(\Delta S)$ is appropriately defined in terms of contributions from Feynman paths. The deviation ΔP is mainly induced by two factors: One comes from contributions not included in the stationary phase approximation, which is used when deriving the semiclassical propagator from Feynman's path integral formulation. Meanwhile, the other is related to the fact that the right hand side of Eq.(9) for classical trajectories does not have an exact Gaussian form. As discussed above, an obvious second-stage non-FGR decay of the LE appears for relatively small values of the dimension N, which correspond to relatively large values of the effective Planck constant. This implies that the above-mentioned first factor plays the major role here, i.e., the deviation is mainly induced by non-semiclassical contributions.

Next, we goes to the Lyapunov regime in the sawtooth model. In this regime, we did not observe a two-stage decay similar to that discussed above in the FGR regime. For large N, as shown in previous work [14], the LE has approximately the semiclassically-predicted Lyapunov decay, with decaying rate given by the Lyapunov exponent. (The sawtooth model has a homogeneous phase space.) With decreasing N, as shown in Fig.3, the decay of the LE gradually deviates from the Lyapunov decay.

In order to quantitatively characterize the above discussed deviation of the exact LE decay from the semiclassical predictions in the Lyapunov regime, we have studied the standard deviation of $x_n \equiv |\ln \overline{M}_e(t = n) - \ln M_{sc}(t = n)|$, where $\overline{M}_e(t)$ denote the exact numerical results. Specifically, we have studied the quantity D,

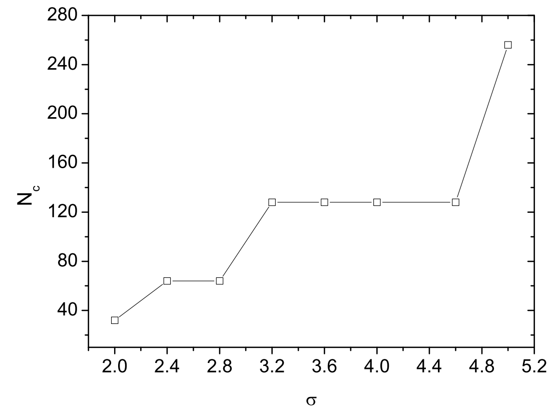

FIG. 5: Variation of N_c with σ in the Lyapunov regime of the sawtooth model with parameter $K = 1.0$.

$$D \equiv \sqrt{\frac{1}{M}\sum_{n=1}^{M}(x_n - \overline{x})^2}, \tag{29}$$

where $\overline{x} = \frac{1}{M}\sum_{n=1}^{M} x_n$ is the average value of x_n. Our numerical simulations show that the value of D remains small for large N and is not small when N is below some value, which we denote by N_c (see Fig.4). That is, the semiclassical prediction for the LE decay works well for N above N_c, but, not well for N below N_c.

The value of N_c was found dependent on the parameter σ, as seen in Fig.5. On average, N_c increases with increasing σ. This dependence may be related to a requirement used in the derivation of the above-mentioned semiclassical predictions for the LE decay, namely, ϵ being small. Indeed, due to the relation $\epsilon = \sigma\hbar_{\text{eff}} = 2\pi\sigma/N$, to keep ϵ small, larger value of σ corresponds to larger value of N.

C. Numerical results in the kicked rotator model

In the kicked rotator model, numerically we found behaviors of the LE more or less similar to those in the sawtooth model discussed in the previous subsection. Specifically, in the FGR regime, when N is not large, we also observed a two-stage decay of the LE; but, in this model, the second-stage decay is faster than the first-stage FGR decay. See Fig.6 for some examples, where the value of $R(E)$ in the FGR decay was computed numerically, making use of Eq.(26).

The kicked rotator does not have a homogeneous phase space, hence, in the Lyapunov regime, the semiclassical prediction for the LE decay is not given by the Lyapunov exponent of the underlying classical dynamics,

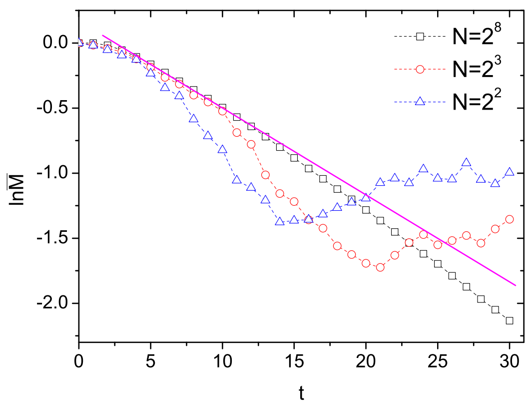

FIG. 6: Same as in Fig.1, for the kicked rotator model with parameters $K = 11.0$, $\sigma = 0.3$ in the FGR regime. The semiclassical prediction of the FGR decay, $e^{-2\sigma^2 R(E)t}$ with $R(E) = 0.375$, is indicated by solid line.

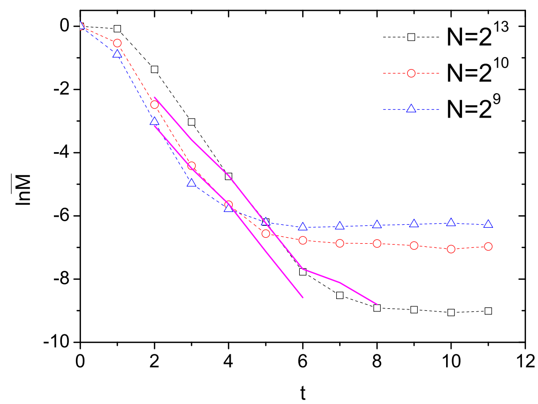

FIG. 7: Similar to Fig.6, but for the Lyapunov regime with parameters $K = 15.0$ and $\sigma = 20.5$ in the kicked rotator model. The semiclassical prediction, given by Eq.(12), is indicated by solid lines.

but, is given by Eq.(12). As expected, only for large N, did we numerically find agreement between predictions of Eq.(12) and the exact LE decay beyond some initial times. For example, as shown in Fig.7, the agreement is good for $N = 2^{13}$ and $t \in [4, 8]$. On the other hand, the agreement is not good for $N \leq 2^9$; for this value of N, the LE approaches its saturation value before the semiclassically-predicted decay in Eq.(12) can be seen.

IV. LE DECAY IN THE VICINITY OF A QUANTUM PHASE TRANSITION

As discussed in Ref.[23], the semiclassical theory can be used in predicting the LE decay in the close neighborhood of those QPTs whose ground levels are infinitely degenerate at the critical points; the closer the controlling parameter λ is to the critical point λ_c, the better the semiclassical theory might work. This is related to the largeness of the density of states near the critical point.

In this section, we study the LE decay close to a critical point of a 1D Ising chain in a transverse field. As discussed below, an effective Planck constant may be introduced in the low energy region in this model, which is inversely proportional to the number N_p of the spins. The system undergoes a QPT at the critical point in the thermodynamic limit $N_p \to \infty$. Here, we are particularly interested in validity/breakdown of the semiclassical prediction of the LE decay in the case of finite N. In this model, H_0 in the definition of the LE in Eq.(1) is taken as $H(\lambda_0)$ and H_1 as $H(\lambda)$. The initial state $|\Psi_0\rangle$ is chosen as the ground state of $H(\lambda_0)$, thus, the LE here is in fact a survival probability.

In this model of 1D Ising chain, the dimensionless Hamiltonian is written as

$$H(\lambda) = -\sum_{i=1}^{N_p} (\sigma_i^z \sigma_{i+1}^z + \lambda \sigma_i^x). \tag{30}$$

The spin-spin interaction intends to force the spins to polarize along the z direction, while the transverse field intends to polarize them along the x direction. The competition of the two interactions results in two critical points, $\lambda_c = \pm 1$, with ferromagnetic phase for $-1 < \lambda < 1$ and paramagnetic phase for $|\lambda| > 1$. Without the loss of generality, we consider the critical point $\lambda_c = 1$.

The above Ising Hamiltonian can be diagonalized by utilizing the Jordan-Wigner and Bogoliubov transformations, giving [24, 40, 41],

$$H(\lambda) = \sum_k e_k (b_k^\dagger b_k - 1/2), \tag{31}$$

where $b_k^\dagger$ and b_k are fermionic creation and annihilation operators, e_k is the corresponding single quasi-particle energy,

$$e_k = 2\sqrt{1 + \lambda^2 - 2\lambda \cos(ka)}, \tag{32}$$

and $k = 2\pi m / a N_p$ with $m = -M, -M+1, \cdots, M$. Here, a is the lattice spacing and $N_p = 2M + 1$.

As discussed in Ref.[23], in the very neighborhood of the critical point with λ sufficiently close to λ_c and for sufficiently large N_p, the low-lying states can be mapped to bosonic modes (labeled by α), with single-particle energies $e_\alpha^b \simeq n_\alpha \delta E$, where $n_\alpha = 1, 2, \ldots$ and $\delta E = 4\pi/N_p$, and, furthermore, the semiclassical theory predicts the FGR decay in Eq.(10) with $R(E)$ given by Eq.(17). The form of the single-particle energy e_α^b suggests that one may introduce an effective Planck constant $\hbar_{\text{eff}}$,

$$\hbar_{\text{eff}} = \delta E = 4\pi/N_p. \tag{33}$$

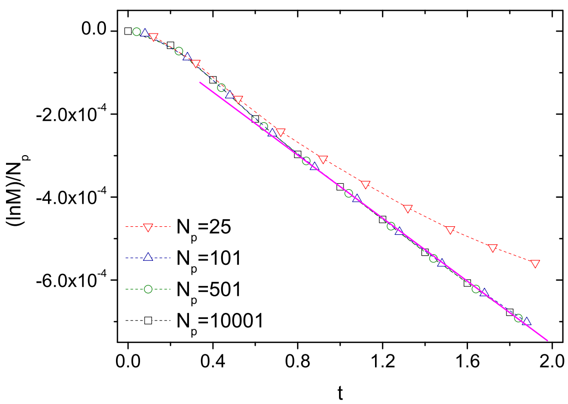

FIG. 8: LE decay of the Ising chain for different values of N_p, with parameters $\lambda_0 = \lambda_c - 4 \times 10^{-2}$, $\lambda = \lambda_c - 10^{-2}$ and $\lambda_c = 1.0$. For large N_p and relatively long times, the LE has an exponential decay as predicted by the semiclassical theory. (A solid straight line is drawn for guiding eyes.) Note also that the LE has a good scaling behavior of $\ln M \propto -N_p t$ for large N_p. For relatively small N_p ($N_p = 25$), the LE has neither the exponential decay nor the scaling behavior.

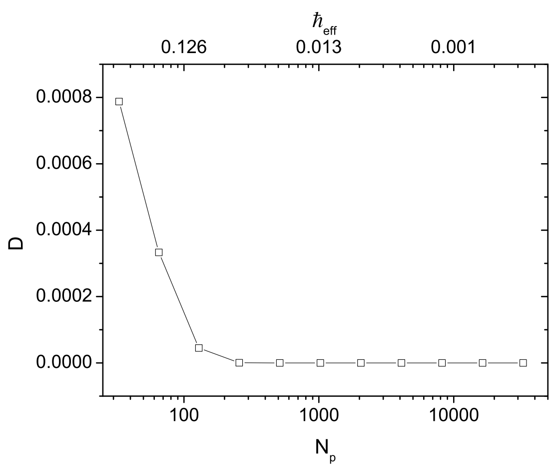

FIG. 9: Variation of the deviation D with the spin number N_p in the Ising model, with parameters $\lambda_0 = \lambda_c - 4 \times 10^{-2}$ and $\lambda = \lambda_c - 10^{-2}$.

Then, the mode energies are written as $e^b_\alpha = \hbar_{\text{eff}}\omega_\alpha$, with $\omega_\alpha \simeq n_\alpha$.

Numerically, we observed the semiclassically-predicted exponential behavior of the LE when N_p is relatively large (see Fig.8). But, for sufficiently small N_p, obvious deviation from the exponential decay appears. We have further calculated the deviation D in Eq.(29) for $x_t = |\ln \overline{M}_e(t) - \ln M_{sc}(t)|$ (see Fig.9), where the slope of $\ln M_{sc}$ was computed in the large N_p limit. It is seen in Fig.9 that, with decreasing N_p, the value of D first remains small, then, begins to increase rapidly around $N_p = 100$, indicating breakdown of the semiclassical prediction.

Furthermore, as seen in Fig.8, when N_p is large, the LE has a good scaling behavior of $\ln M(t) \sim N_p$. To understand this phenomenon, we note that here the perturbation ϵV in the definition of the LE takes the form of $\epsilon = (\lambda_0 - \lambda)$ and

$$V = \sum_{i=1}^{N_p} \sigma_i^x. \tag{34}$$

Hence, according to Eq.(17), the quantity $R(E)$ is given by the square of the summation of N_p terms with mean zero, each of which is a time integration of $\epsilon(\sigma_i^x - \overline{\sigma_i^x})$. As discussed in Ref.[23], in the low energy region and near the critical point of this model, the system has a classical counterpart with many degrees of freedom, hence, possesses a classical motion with many frequencies. This implies that the time integrations mentioned above can usually be regarded as being uncorrelated. Then, for large N_p, the quantity $R(E)$ is approximately proportional to N_p, hence, $\ln M(t) \sim N_p$.

V. CONCLUSIONS

In this paper, we have studied the change from validity to breakdown of semiclassical predictions for the LE decay in several models, when the effective Planck constants are increased and the systems move from the semiclassical region to the deep quantum region. Our numerical results show that some semiclassical predictions for the LE decay work well even in the deep quantum region.

In particular, in the FGR regime of quantum chaotic systems, there is always some time interval within which the LE follows the FGR decay; but, the length of this time interval decreases with increasing effective Planck constant. Beyond this time interval, deviation from the FGR decay appears, which is induced by non-semiclassical contributions. In the Lyapunov regime, the semiclassical prediction remains valid when the effective Planck constant is not quite small, but, it becomes invalid for sufficiently small effective Planck constant.

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